Introduction to Complexity and Applied Complexity, Spring 2021

Module 3 — Difference Equations, Nonlinear Dynamics and Chaos

Notes by Sav Sidorov

Readings

- N/A

On Models

One of the important things to keep in mind when modeling the world is to know the limitations of your models, and how readily "scientific" assumptions break down. There's a long history of fields like "scientific" forestry, or "scientific" management failing miserably. Why? Because what they did was build a model of the world and then used the model as a stand in for reality. Remember, the map is not the territory. Even worse, models are sometimes treated as "the platonic ideal" that the real world should aspire to. This is where you get notions like "let's "fix" the real world by making it conform to the model." That way of thinking has been a complete disaster. Models can be dangerous if you use them that way.

The right way to use models is as a small component of a larger approach/strategy of making sense of a phenomena, internalizing the fact that you can't formalize some parts of the system.

When used in this way, taking into account complex systems thinking, models can aid in avoiding catastrophe. That's probably their most important application.

Trends of the 21st Century

The technology of the 21st Century, in all likelihood, will not be like the technology of the 20th century. The 20th century was largely defined by gadgets — cars, airplanes, home appliances. Discrete *things* that function on their own. If you look at the aesthetic and societal trends, you'll notice this as well (retrofuturism, modernism, art deco) — a kind of compartmentalization of things, like the different scientific fields we talked about with that XKCD comic in Module 1.

The 21st Century, on the other hand, is all about how systems are organized. How do things fit together? What's the bigger picture? What are the interactions and interrelations between the components? To the extent that it'll be about gadgets, it'll be about gadgets that enable different kinds of organization and interaction between agents in the system.

An Abundance of Uncertainty

In this part of the course, we just want to get a sense of how little it takes to blow up all of our scientific methodologies — our empirical methodologies — when trying to look at a system and understand what's going on. When we look at models (like the difference equations, for instance), we have access to the ground truth, so to speak. We know everything about them. We know what the rules are, what the states are — everything. And yet, what we'll see in this Module, is that even in a system where we have all the information, we can't necessarily know what's going on or what's going to happen. And in the real world, of course, you essentially never have even this kind of knowledge. All of this sets bounds on our ability to use our mental processes to project what's going to happen and predict the world.

This, of course, is what Nassim Taleb's life work is all about — the problem of uncertainty, and decision making under that uncertainty. What might we do about this problem? The naive reaction might be to say that if something's uncertain, you can't say anything about it — there's nothing you can do. But that assumption turns out to be false. Once you know that there's uncertainty in the system, you can, for one, start classifying different kinds of uncertainty, and develop approaches for dealing with them.

Departing from Newton

One thing we looked at in the previous Module was a system with input. The input gives the system some indication of which mapping it's going to behave according to. This is a very different conception than the Newtonian forcing conception, for example. Seeing a red octagon and hitting the brakes of your car as a result is not well explained in terms of forces acting on the system. It's much better explained as information transfer — from the sign, to your perceptual organs, brain, and finally to your motor organs.

Growth Equations

A growth equations is anything stated in the form:

$$x\left[t+1\right]=Ax\left[t\right]$$

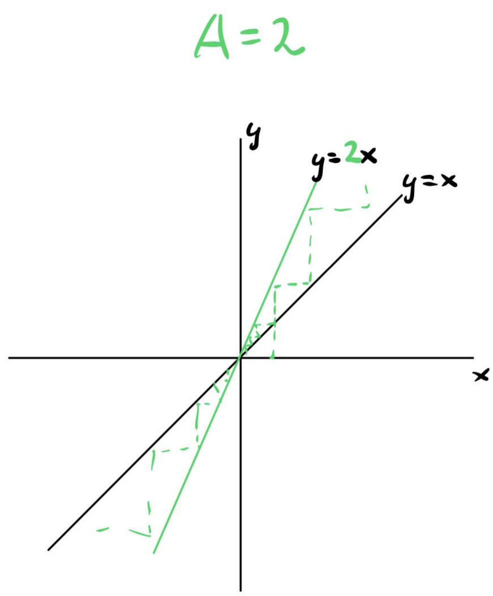
Where A is a parameter of this equation. We can vary this parameter and see how the behavior of our system changes. The reason it's called a growth equation is because it looks like biological, multiplicative growth — it scales any given input.

If we take A=2, and start our *x[t]* at 1, we will get the following pattern by continuously recursing on our function (taking the output and plugging it back as an input):

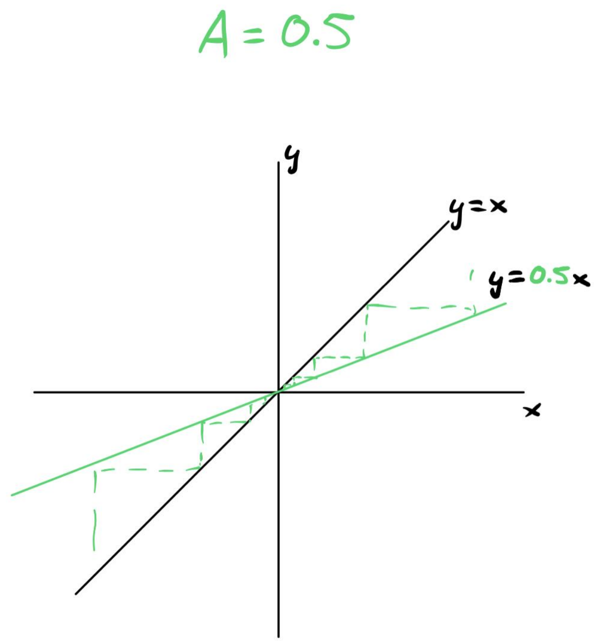
$$1, \; 2, \; 4, \; 8, \; 16, \ldots$$

Wherever our function crosses the identity line (*x = y*), we have a fixed point. In the case of our growth equation, this would be *x = y = 0*.

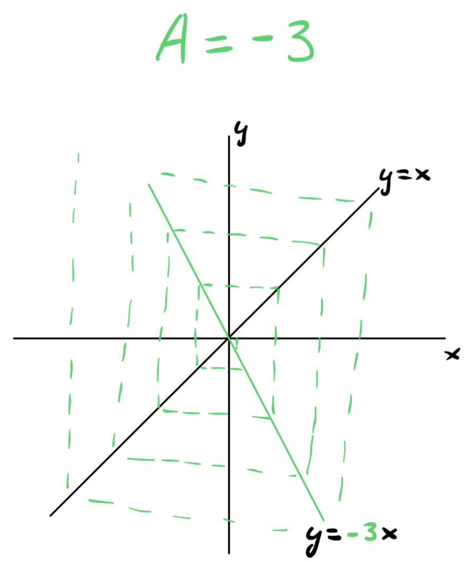
Additionally, the slope tells us what kind of fixed point it is. In this equation, 0 is an unstable fixed point. Any input slightly above zero explodes the function, if A is greater than 1.



Now, if $0 < A < 1$, then 0 would be a **stable fixed point**, and no matter how large in input, it will always be brought down to zero.



We can also have a negative slope. For example:



This is often associated with an oscillation that’s growing.

One thing to note, these are all **linear systems**. That is to say, the phase plot is linear. You can either get exponential growth (**A = 2**) out of that, or exponential decay (**A = 0.5**).

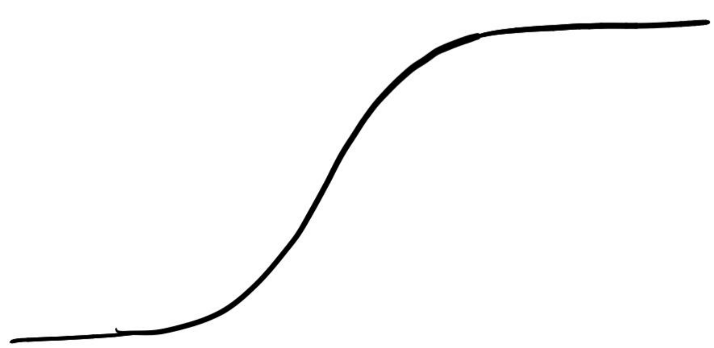
Or some version of growth but on the negative side, where the sign oscillates as the system grows (**A = —3**).

In the real world, there is a finite amount of resources. Because of that, things can't grow forever. Things that grow tend to reach this point of **saturation**. That accounts for the historical development of something called the **logistic map**.

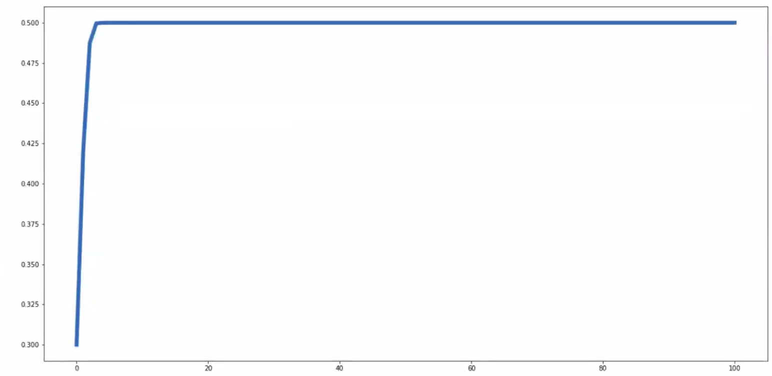
Let's take an example equation — just like our growth equation, but with an additional term:

$$x\left[t+1\right]=Ax\left[t\right]\left(1-x\left[t\right]\right);x\in\left[0,1\right]$$

The notation on the right means that *x* is **an element of** the set 0 to 1. The idea behind this equation is to model growth with this point of saturation. If *x* is very small, it grows rapidly, but as *x* increases to 1, the function flatlines.

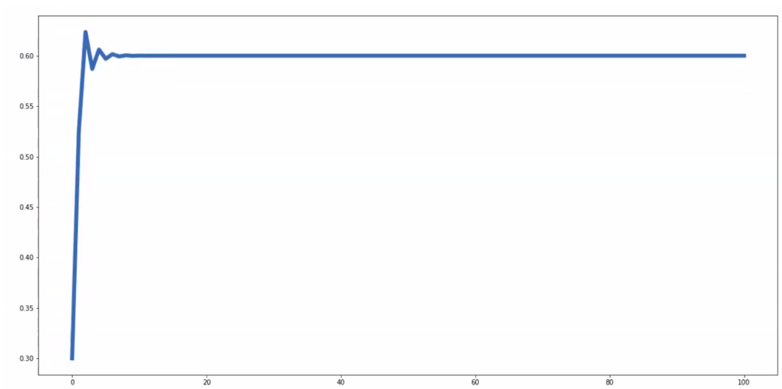


Here, for example, is a graph of this function at *A* = 2, with the initial condition x_0 = 0.3:



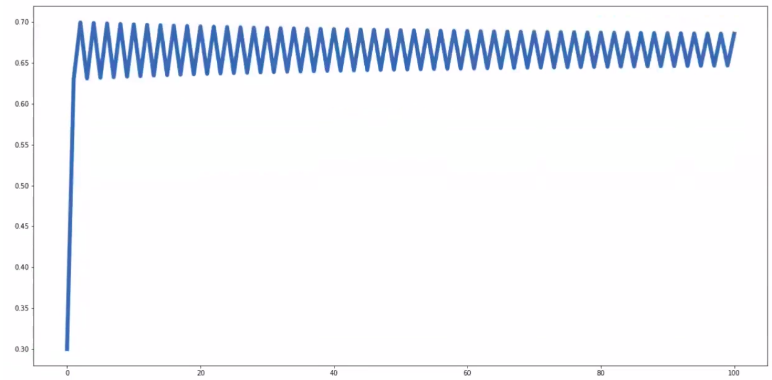
A fixed point attractor. Very much as expected.

But what happens as we sweep the parameter? That's how we should think about probing dynamical systems — it's not just about the behavior at given parameters, but also about how the behavior changes as we adjust the parameters. At $A = 2.5$, there's a bit of an oscillation before things reach the saturation point:

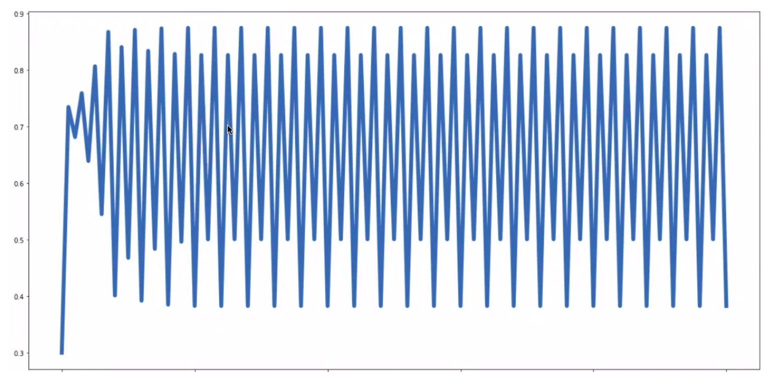


This is a phenomenon called **dampening**. An oscillating system is **overdamped** if the damping is enough to extinguish the oscillation. A system is **underdamped** if the oscillation is getting squashed, but not enough to put it out completely. A system is **critically damped** when there's *juuust* enough damping to quell the oscillation.

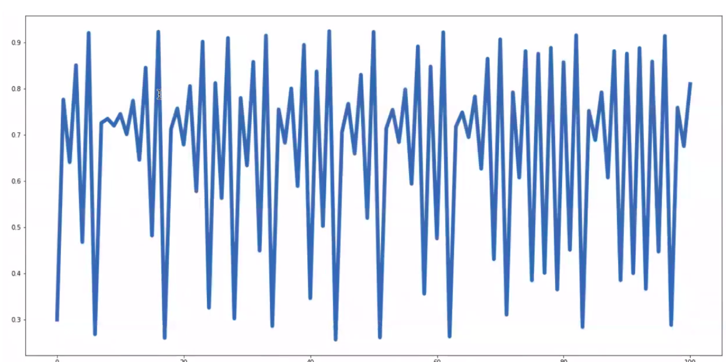
As we increase our A to 3, we see that the oscillation does not get dampened. The system stabilizes around this oscillation — oscillating between two points.



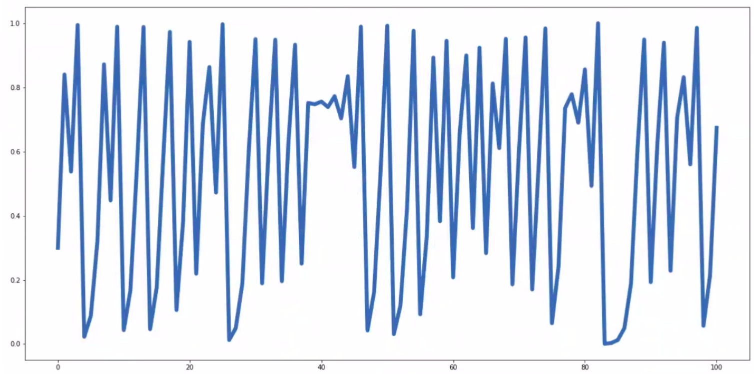
We can also have a four-period oscillation — going between four fixed points:



As we keep increasing A, we'll notice that the oscillation stops being periodic…

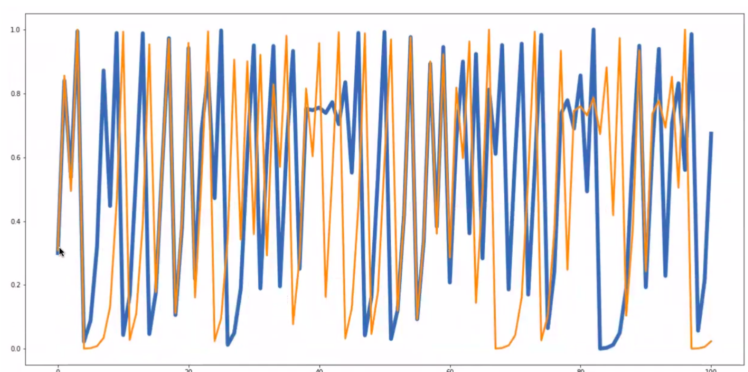


And becomes chaotic ($A = 4$):



Notice that this equation is easy to look at — it's easy to see what's going on from a procedural standpoint. But as you iterate on it, you get dynamics that look *as if* they're produced by random numbers. Of course we know they're not random, they're deterministic. But that's what they end up looking like.

Of course, if you change the initial conditions even slightly, you get wildly different behavior (here we have two runs, one in blue and one in orange):



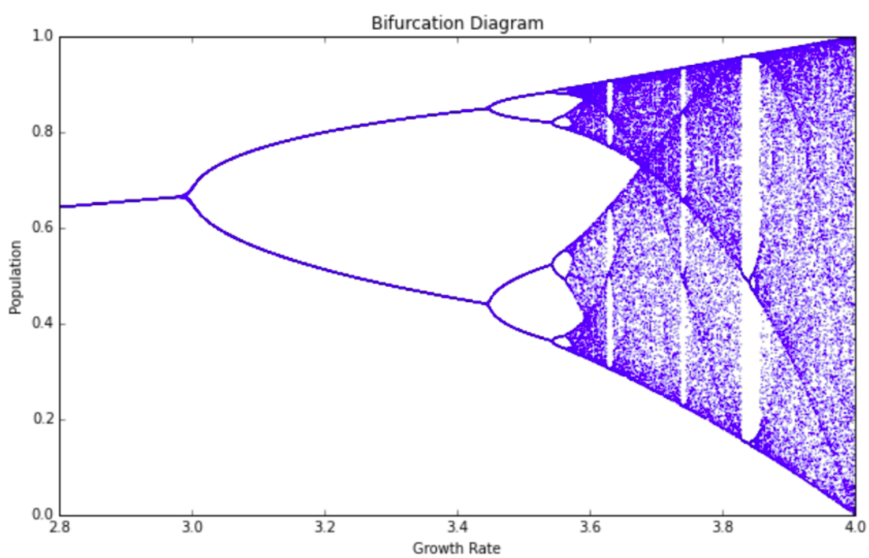
We can say that **chaotic dynamics** are *extremely* sensitive to **initial conditions**.

Again, there's nothing random going on here. But the oscillations behave kind of randomly, jumping all over the place to different values instead of repeating their patterns. And indeed, the smallest amount of error means that you can only predict the trajectory for a short amount of time.

The takeaway here is that you shouldn’t spend all your time trying to know the state of the system with greater and greater precision so that we can predict better. This is a temptation a lot of people fall into. “We just need better data, right?” “We just need to know a bit more about what's going on, and then we'll be able to predict.” No — if you happen to find a system that has a stable, fixed point equilibrium, then you don't need much resolution at all to know what's going to happen. If you happen to have a chaotic system, then *no amount of resolution increasing will ever get you there*.

Bifurcation Diagrams

Bifurcation diagrams are another important way to look at the behavior of dynamical systems. Here we’ve constructed a bifurcation diagram of our logistic map:



On the x-axis, we have the growth rate parameter, A, going from 2.8 to 4. On the y-axis, we’re showing the long-term stability (or instability) of the population in our logistic map. So in other words, the y-axis is the same one we had before. We can see that between A = 2.8 and A = 3, we get a single value — which is what we saw when we graphed $A = 2$. As we increase A, we begin to see that oscillation between values — first two, then four, until eventually the system becomes chaotic.

Chaotic Dynamics

To sum up, what do we mean when we say that a system displays chaotic dynamics?

1. We saw that we can have a totally deterministic system give rise to behavior that looks random. We get what are called **deterministic chaotic dynamics**.

2. There is a great sensitivity to initial conditions. This makes it very hard to predict a system past some short timeframe of observation; **there's a finite horizon of predictability**.

3. And of course, all of this is **a major monkey wrench in the Newtonian expectations of predictability**. Laplace's Demon turns out to be wrong, at least in a practical sense — if you have chaotic dynamics and a finite resolution, you can't measure down to infinite detail. All of that predictability goes out the window, even though you've retained determinism.

Here's a quote relevant to this — from the 2001 IPCC report on climate:

> *"The climate system is a **coupled non-linear chaotic system**, and therefore the long-term prediction of future exact climate states is **not possible.**"* IPCC, 2001

20 years later, and nothing has changed about the truth of this statement. It is true. Of course, what has changed is all the politics around it. Hopefully this helps to indicate how we've infected science with scientism — a big problem we have to deal with. Developing some sensitivity to the difference between actual science and scientism is important. This does not imply that there is nothing to worry about with regard to climate. We depend on climate — we depend on there being some kind of regularity to our weather systems. But we can be honest and say, "when you make these long term predictions, you're full of shit." We know that the fundamentals of science say that you

can't actually make claims in this manner. But at the same time, we should become much more sensitive to taking care of our environment.

Nonlinear Dynamics

Let’s say a few words about the nonlinear dynamics of systems.

For one, we obviously start to see a much richer set of possible behaviors than in linear dynamics. In our difference equation, we saw some oscillating behavior. Yet if we look at continuous representations of discrete time equations, linear dynamics become even more boring. Continuous linear systems are usually what people mean when they talk about linear dynamics, where you pretty much only have a fixed point. And it can either be stable or unstable, and you can't get oscillations. To get those oscillations you need at least a two dimensional system in the continuous case.

Nonlinear systems also exhibit a kind of spontaneous, non-forced behavior. They operate under their own internal organization and dynamical laws. Unlike the linear systems we looked at in this Module, they are not decomposable (not reducible).

Let’s close with a quote from Ashby.

> *“Cybernetics might, in fact, be defined as the study of systems that are open to energy but closed to information and control — systems that are ‘information-tight’.”*

When he says “closed to information and control”, he's simply talking about the kind of closure we discussed earlier — the fact that these mathematical functions, these maps, always produce some state of the system that the function can then operate on again, enabling systems to evolve indefinitely. When he says they're “open to energy”, he’s saying that the kinds of behaviors we're looking at don’t just evolve to a static equilibrium and stop — they can oscillate indefinitely, their oscillations can grow and they can become chaotic. These kinds of systems don't exist at thermodynamic equilibrium — they have to have some flow of energy into them.

This is a really important idea for complex systems generally: open systems (open in the sense of thermodynamics) take in energy and matter, and they expel energy and matter. And in that flow, they're able to produce behaviors that appear to be against certain known physical laws. Specifically, the second law of thermodynamics, where everything evolves toward increasing entropy, appears to be violated. These nonlinear systems do not evolve toward increasing entropy. That’s because of this openness to a bidirectional flow of energy and matter. The second law of thermodynamics depends on the assumption that you're talking about isolated systems. So that's the sense in which these systems are not isolated — they're open to flows of energy.